

CONDITIONAL EXPECTATIONS IN C^* -CROSSED PRODUCTS

BY

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ABSTRACT. Let (A, G, α) be a C^* -dynamical system. Let B be a C^* -subalgebra of A and P be a conditional expectation of A onto B such that $\alpha_t P = P \alpha_t$ for each $t \in G$. Then it is proved that there exists a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G, B, \alpha)$. In particular, if G is amenable and A is unital, then there always exists a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G)$. Some related results are also obtained.

1. Introduction. Recently Anantharaman-Delaroche [1], [2] investigated the existence of a conditional expectation of a W^* -crossed product $W^*(G, M, \alpha)$ onto a W^* -crossed product $W^*(G, N, \alpha)$ under appropriate conditions, where N is a von Neumann subalgebra of a von Neumann algebra M .

In this paper corresponding results for C^* -crossed products are studied and analogous results are obtained. In particular, if a C^* -algebra A is unital and G is amenable, then the existence of a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G)$ is shown.

2. Projections in C^* -algebras. Let A be a C^* -algebra, and let $M_n(A)$ be the C^* -algebra of $n \times n$ matrices $M = [a_{ij}]$ with entries a_{ij} in A (cf. Paschke [10, Appendix], Takesaki [17, IV. §3]).

LEMMA 2.1 (CF. PASCHKE [10, PROPOSITION 6.1], TAKESAKI [17, LEMMA IV.3.2]). *An element $M = [a_{ij}]$ of $M_n(A)$ is positive if and only if $\sum_{i,j} x_i^* a_{ij} x_j \geq 0$ in A for any $x_1, \dots, x_n \in A$.*

Let B be a C^* -subalgebra of A . A bounded linear map $P: A \rightarrow B$ is called a conditional expectation if P has the following properties (cf. Umegaki [19]):

- (i) P is an onto projection of norm one, that is, $P^2 = P$ and $\|P\| = 1$;
- (ii) P is positive, that is, for any $x \in A$, $P(x^*x) \geq 0$;
- (iii) for any $x \in A, y, z \in B$, $P(yxz) = yP(x)z$;
- (iv) for any $x \in A$, $P(x^*)P(x) \leq P(x^*x)$.

If $P: A \rightarrow B$ is a conditional expectation, then, by (ii), $P(x^*) = P(x)^*$ for each $x \in A$. Tomiyama [18] proved that if $P: A \rightarrow B$ is an onto projection of norm one (that is P satisfies condition (i)), then P becomes a conditional expectation (cf. Takesaki [17, Theorem III.3.4]). In fact, in this case P satisfies conditions (ii) and (iv) above and

- (iii)' $P(xy) = P(x)y$ and $P(yx) = yP(x)$ for every $x \in A, y \in B$.

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PROPOSITION 2.2. *Let $P: A \rightarrow B$ be a conditional expectation. Then, for any positive integer n and any $x_1, \dots, x_n \in A$, the $n \times n$ matrix $M = [P(x_i^* x_j) - P(x_i^*)P(x_j)]$ of $M_n(B)$ is positive.*

PROOF. For each $y_1, \dots, y_n \in B$, $x = \sum_i x_i y_i$ is in A . By condition (iv) we have $P(x^*)P(x) \leq P(x^* x)$; hence

$$\sum_{i,j} y_i^* P(x_i^*) P(x_j) y_j \leq \sum_{i,j} y_i^* P(x_i^* x_j) y_j.$$

By Lemma 2.1, the matrix $M = [P(x_i^* x_j) - P(x_i^*)P(x_j)]$ is positive in $M_n(B)$.

COROLLARY 2.3 (NAKAMURA, TAKESAKI AND UMEGAKI [9], cf. STØRMER [14, THEOREM 4.1]). *Let A , B and P be as above. Then, for each positive integer n and each $x_1, \dots, x_n \in A$, $M = [P(x_i^* x_j)]$ is a positive element of $M_n(B)$, that is, P is a completely positive map (cf. Takesaki [17, IV. §3]).*

3. Projections in C^* -crossed products. Let G be a locally compact group and dt be the left Haar measure on G . Let A^* be the dual space of a C^* -algebra A and $\langle \cdot, \cdot \rangle$ be the duality pairing between A and A^* . Denote by $\text{Aut}(A)$ the $*$ -automorphism group of A . We suppose that $\alpha: G \rightarrow \text{Aut}(A)$ is a strongly continuous homomorphism. Then (A, G, α) is called a C^* -dynamical system.

Let $L^1(G, A)$ be the set of all (equivalence classes of) A -valued Bochner integrable functions on G with respect to dt . $L^1(G, A)$ is a Banach $*$ -algebra with an approximate identity whose multiplication, involution and norm are respectively defined by

$$(xy)(t) = \int x(s) \alpha_s(y(s^{-1}t)) ds, \quad (x^*)(t) = \Delta(t)^{-1} \alpha_t(x(t^{-1}))^*,$$

$$\|x\|_1 = \int \|x(t)\| dt$$

for each $x, y \in L^1(G, A)$ and $t \in G$, where Δ is the modular function of G (Doplicher, Kastler and Robinson [5, §§II, III]; cf. Bratteli and Robinson [3, §2.7.1], Pedersen [11, §7.6]). We denote by $C^*(G, A, \alpha)$ the enveloping C^* -algebra of $L^1(G, A)$ (Doplicher, Kastler and Robinson [5, §IV]; cf. Bratteli and Robinson [3, §2.7.1], Pedersen [11, 7.6.5]). $C^*(G, A, \alpha)$ is called the C^* -crossed product (or the covariance algebra) of (A, G, α) . We also denote by $C^*(G)$ the group C^* -algebra of G (cf. Dixmier [4, 13.9.1], Pedersen [11, 7.1.5]). $C^*(G)$ is nothing but $C^*(G, \mathbb{C}, \alpha_0)$, where \mathbb{C} is the complex numbers and $\alpha_0: G \rightarrow \text{Aut}(\mathbb{C})$ is the unique trivial homomorphism.

Now let P be a conditional expectation of A onto a C^* -subalgebra B of A . Assume that $\alpha_t P = P \alpha_t$ for every $t \in G$. Then for each $t \in G$, $\alpha_t(B) \subset B$, hence α_t may also be considered as a $*$ -automorphism of B .

PROPOSITION 3.1. *Let (A, G, α) be a C^* -dynamical system, B be a C^* -subalgebra of A , and P be a conditional expectation of A onto B . Suppose that for any $t \in G$, $\alpha_t P = P \alpha_t$. Then $C^*(G, B, \alpha)$ is a C^* -subalgebra of $C^*(G, A, \alpha)$.*

PROOF. If $y \in L^1(G, B)$, then $y \in L^1(G, A)$; thus it is sufficient to show that the norm $\|y\|_B$ of y in $C^*(G, B, \alpha)$ is equal to the norm $\|y\|_A$ of y in $C^*(G, A, \alpha)$. The inequality $\|y\|_A \leq \|y\|_B$ is clear from the definition. We must prove that $\|y\|_A \geq \|y\|_B$. Let Ψ be a (continuous) positive linear form on $L^1(G, B)$ with $\|\Psi\| \leq 1$. Then Ψ extends to a positive linear form on $C^*(G, B, \alpha)$ (cf. Dixmier [4, 2.7.5]). We use the same symbol Ψ for the extended linear form. To this Ψ , there corresponds a norm continuous positive definite function $\psi: G \rightarrow B^*$ by Pedersen [11, 7.6.7, 7.6.8]. Define $\phi: G \rightarrow A^*$ by $\langle a, \phi(t) \rangle = \langle P(a), \psi(t) \rangle$ ($a \in A, t \in G$). Then ϕ is a norm continuous positive definite function. In fact, for any positive integer n , any $a_1, \dots, a_n \in A$ and any $t_1, \dots, t_n \in G$, the $n \times n$ matrix $M = [P(a_i^* a_j)]$ is a positive element in $M_n(B)$ by Corollary 2.3. Thus $M = N^* N$ for some $N = [b_{ij}] \in M_n(B)$, that is, $P(a_i^* a_j) = \sum_k b_{ki}^* b_{kj}$ ($1 \leq i, j \leq n$). Since ψ is positive definite, it follows that

$$\begin{aligned} \sum_{i,j} \langle \alpha_{t_i^{-1}}(a_i^* a_j), \phi(t_i^{-1} t_j) \rangle &= \sum_{i,j} \langle P(\alpha_{t_i^{-1}}(a_i^* a_j)), \psi(t_i^{-1} t_j) \rangle \\ &= \sum_{i,j} \langle \alpha_{t_i^{-1}}(P(a_i^* a_j)), \psi(t_i^{-1} t_j) \rangle \\ &= \sum_{i,j} \left\langle \alpha_{t_i^{-1}} \left(\sum_k b_{ki}^* b_{kj} \right), \psi(t_i^{-1} t_j) \right\rangle \\ &= \sum_k \sum_{i,j} \langle \alpha_{t_i^{-1}}(b_{ki}^* b_{kj}), \psi(t_i^{-1} t_j) \rangle \geq 0. \end{aligned}$$

Hence ϕ is positive definite. Let Φ be a positive linear form on $C^*(G, A, \alpha)$ corresponding to ϕ by Pedersen [11, 7.6.7, 7.6.8]. Then $\|\Phi\| = \|\phi(e)\| = \|\psi(e)\| = \|\Psi\| \leq 1$, where e is the identity of G (cf. Pedersen [11, 7.6.7]), and

$$\Phi(y^* y) = \int \langle (y^* y)(t), \phi(t) \rangle dt = \int \langle (y^* y)(t), \psi(t) \rangle dt = \Psi(y^* y).$$

This implies that $\|y\|_B \leq \|y\|_A$. Therefore, $\|y\|_B = \|y\|_A$ for any $y \in L^1(G, B)$, and $C^*(G, B, \alpha)$ is a C^* -subalgebra of $C^*(G, A, \alpha)$. (In the sequel we denote by $\|x\|$ the norm of $x \in L^1(G, A)$ in $C^*(G, A, \alpha)$.)

THEOREM 3.2. *Let (A, G, α) , B and P be as in Proposition 3.1. Then there exists a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G, B, \alpha)$.*

PROOF. Define $Q: L^1(G, A) \rightarrow L^1(G, B)$ by $(Q(x))(t) = P(x(t))$ ($x \in L^1(G, A)$, $t \in G$). Then Q is a linear map and $Q(y) = y$ for every $y \in L^1(G, B)$. Take a positive linear form Ψ on $L^1(G, B)$ with $\|\Psi\| \leq 1$. As in the proof of Proposition 3.1, to this Ψ there correspond a norm continuous positive definite function $\psi: G \rightarrow B^*$, a norm continuous positive definite function $\phi: G \rightarrow A^*$ and a positive linear form Φ on $C^*(G, A, \alpha)$ with $\|\Phi\| \leq 1$. Let $K(G, A)$ be the set of all A -valued

continuous functions on G with compact support. Then for $x \in K(G, A)$,

$$\begin{aligned}\Psi(Q(x)*Q(x)) &= \int \langle (Q(x)*Q(x))(t), \psi(t) \rangle dt \\ &= \int \int \Delta(s^{-1}) \langle \alpha_s(P(x(s^{-1})*)P(x(s^{-1}t))), \psi(t) \rangle ds dt \\ &= \int \int \langle \alpha_{s^{-1}}(P(x(s)*)P(x(t))), \psi(s^{-1}t) \rangle ds dt\end{aligned}$$

and

$$\begin{aligned}\Phi(x*x) &= \int \langle (x*x)(t), \phi(t) \rangle dt \\ &= \int \int \Delta(s^{-1}) \langle \alpha_s(P(x(s^{-1})*)x(s^{-1}t)), \psi(t) \rangle ds dt \\ &= \int \int \langle \alpha_{s^{-1}}(P(x(s)*)x(t)), \psi(s^{-1}t) \rangle ds dt.\end{aligned}$$

The function

$$(s, t) \rightarrow \langle \alpha_{s^{-1}}(P(x(s)*)P(x(t))), \psi(s^{-1}t) \rangle$$

on $G \times G$ is continuous and of compact support S_1 . Similarly the function

$$(s, t) \rightarrow \langle \alpha_{s^{-1}}(P(x(s)*)x(t)), \psi(s^{-1}t) \rangle$$

on $G \times G$ is continuous and of compact support S_2 . Then S_1 and S_2 are contained in a set $K \times K$ for some compact subset K of G . The measure on K induced by the left Haar measure dt is a finite measure, hence it is the weak*-limit of positive measures m_i of finite support. Then $m_i = \sum_j c_j \delta_{t_j}$ for some positive numbers c_1, \dots, c_n and $t_1, \dots, t_n \in G$, where δ_t is the point measure at $t \in G$. We have

$$\begin{aligned}\int \int \langle \alpha_{s^{-1}}(P(x(s)*)P(x(t))), \psi(s^{-1}t) \rangle dm_i(s) dm_i(t) \\ = \sum_{i,j} \langle \alpha_{t_i^{-1}}(P(x(t_i)*)P(x(t_j))), \psi(t_i^{-1}t_j) \rangle c_i c_j\end{aligned}$$

and

$$\begin{aligned}\int \int \langle \alpha_{s^{-1}}(P(x(s)*)x(t)), \psi(s^{-1}t) \rangle dm_i(s) dm_i(t) \\ = \sum_{i,j} \langle \alpha_{t_i^{-1}}(P(x(t_i)*)x(t_j)), \psi(t_i^{-1}t_j) \rangle c_i c_j.\end{aligned}$$

By Proposition 2.2, $M = [c_i c_j P(x(t_i)*)x(t_j) - c_i c_j P(x(t_i)*)P(x(t_j))]$ is positive in $M_n(B)$. Thus, $M = N^*N$ for some $N = [b_{ij}] \in M_n(B)$. It follows that

$$\begin{aligned}\sum_{i,j} \langle \alpha_{t_i^{-1}}(P(x(t_i)*)x(t_j)), \psi(t_i^{-1}t_j) \rangle c_i c_j - \sum_{i,j} \langle \alpha_{t_i^{-1}}(P(x(t_i)*)P(x(t_j))), \psi(t_i^{-1}t_j) \rangle c_i c_j \\ = \sum_{i,j} \left\langle \alpha_{t_i^{-1}} \left(\sum_k b_{ki}^* b_{kj} \right), \psi(t_i^{-1}t_j) \right\rangle = \sum_k \sum_{i,j} \langle \alpha_{t_i^{-1}}(b_{ki}^* b_{kj}), \psi(t_i^{-1}t_j) \rangle > 0.\end{aligned}$$

Therefore

$$\begin{aligned} & \int \int \langle \alpha_{s^{-1}}(P(x(s)^*P(x(t))), \psi(s^{-1}t)) \rangle dm_t(s) dm_t(t) \\ & \leq \int \int \langle \alpha_{s^{-1}}(P(x(s)^*x(t))), \psi(s^{-1}t)) \rangle dm_t(s) dm_t(t) \end{aligned}$$

and, in the limit,

$$\begin{aligned} & \int \int \langle \alpha_{s^{-1}}(P(x(s)^*P(x(t))), \psi(s^{-1}t)) \rangle ds dt \\ & \leq \int \int \langle \alpha_{s^{-1}}(P(x(s)^*x(t))), \psi(s^{-1}t)) \rangle ds dt. \end{aligned}$$

We have $\Psi(Q(x)^*Q(x)) \leq \Phi(x^*x)$. Since $K(G, A)$ is dense in $L^1(G, A)$, the above inequality holds for any $x \in L^1(G, A)$. This implies that $\|Q(x)\| \leq \|x\|$ for every $x \in L^1(G, A)$, and Q extends to a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G, B, \alpha)$.

Let ϕ be a state of A . If $\langle \alpha_t(a), \phi \rangle = \langle a, \phi \rangle$ for any $a \in A, t \in G$, then ϕ is said to be an α -invariant state. Let $CB(G)$ be the Banach algebra of all complex-valued bounded continuous functions on G with supremum norm. For each state ϕ of A and $a \in A$, define $\phi_a \in CB(G)$ by $(\phi_a)(t) = \langle \alpha_t(a), \phi \rangle$. If A is unital and G is amenable, then for any (left and right) invariant mean m (cf. Greenleaf [7, p. 29], Pedersen [11, 7.3.3]), the state ϕ_m of A defined by $\langle a, \phi_m \rangle = m(\phi_a)$ is α -invariant (cf. Emch [6, p. 173]). Notice that, if G is amenable, the reduced C^* -crossed product $C_r^*(G, A, \alpha)$ (Zeller-Meier [20, Définition 4.6 (for G discrete)], Takai [15]; cf. Pedersen [11, 7.7.4]) is equal to the C^* -crossed product $C^*(G, A, \alpha)$ for any C^* -dynamical system (A, G, α) (Zeller-Meier [20, Théorème 5.1 (for G discrete)], Takai [15, Proposition 2.2]; cf. Pedersen [11, 7.7.7]). It is known that G is amenable if and only if $C^*(G) = C_r^*(G)$ (cf. Dixmier [4, §18.3], Greenleaf [7, §3.5], Pedersen [11, §7.3]).

COROLLARY 3.3. *Let (A, G, α) be a C^* -dynamical system. Suppose that A is unital and has an α -invariant state. Then there exists a conditional expectation of $C^*(G, A, \alpha)$ onto its C^* -subalgebra $C^*(G)$.*

PROOF. Let 1 be the identity of A and identify C with $C1$. Let ϕ be an α -invariant state of A . Define $P: A \rightarrow C1$ by $P(a) = \phi(a)1$ ($a \in A$). Then P is a conditional expectation and, since ϕ is α -invariant, $P\alpha_t = \alpha_t P$ for every $t \in G$. By Theorem 3.2 there exists a conditional expectation of $C^*(G, A, \alpha)$ onto the C^* -subalgebra $C^*(G)$ of $C^*(G, A, \alpha)$.

COROLLARY 3.4. *Let (A, G, α) be a C^* -dynamical system with A unital. If G is amenable, then there exists a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G)$.*

Now we consider the case where G is abelian. Denote by \hat{G} the dual group of G . For each $\sigma \in \hat{G}$, define $\hat{\alpha}_\sigma: L^1(G, A) \rightarrow L^1(G, A)$ by

$$(\hat{\alpha}_\sigma(x))(t) = \overline{(t, \sigma)} x(t) \quad (x \in L^1(G, A), t \in G),$$

where (t, σ) is the value of the character σ at t . Then $\hat{\alpha}_\sigma$ can be extended to a $*$ -automorphism of $C^*(G, A, \alpha)$ and, denoting it by the same symbol $\hat{\alpha}_\sigma$, $\hat{\alpha}: G \rightarrow \text{Aut}(C^*(G, A, \alpha))$ is shown to be a strongly continuous homomorphism (Takai [15, pp. 30–31]; cf. Pedersen [11, 7.8.3]). $(C^*(G, A, \alpha), \hat{G}, \hat{\alpha})$ is called the dual C^* -dynamical system of (A, G, α) .

COROLLARY 3.5. *Let (A, G, α) be a C^* -dynamical system with A unital and G abelian. Then there exists a conditional expectation of $C^*(\hat{G}, C^*(G, A, \alpha), \hat{\alpha})$ onto $C^*(\hat{G}, C^*(G), \hat{\alpha})$.*

PROOF. By Corollary 3.4 there exists a conditional expectation P of $C^*(G, A, \alpha)$ onto $C^*(G)$. In view of the construction of P , it is easy to see that $\hat{\alpha}_\sigma P = P\hat{\alpha}_\sigma$ for all $\sigma \in \hat{G}$. Hence, by Theorem 3.2 there exists a conditional expectation of $C^*(\hat{G}, C^*(G, A, \alpha), \hat{\alpha})$ onto $C^*(\hat{G}, C^*(G), \hat{\alpha})$.

4. Projections in C^* -crossed products with discrete groups. We now treat the case where G is discrete. The following theorem is essentially due to Zeller-Meier [20].

THEOREM 4.1. *Let (A, G, α) be a C^* -dynamical system with G discrete. Then*

- (i) *there exists a conditional expectation of $C_r^*(G, A, \alpha)$ onto A , and*
- (ii) *there exists a conditional expectation of $C^*(G, A, \alpha)$ onto A .*

PROOF. (i) Since G is discrete, the Haar measure dt on G is a counting measure, that is, for any finite subset F of G , the measure of F is the number of elements in F . By the correspondence $a \rightarrow \delta_e a$ ($a \in A$), A may be considered as a C^* -subalgebra of $C_r^*(G, A, \alpha)$ [20, p. 171], where δ_e is the characteristic function of $\{e\}$, that is, $\delta_e(t) = 1$ if $t = e$, or 0 if $t \neq e$. Define $P: L^1(G, A) \rightarrow A$ by $P(x) = x(e)$ ($x \in L^1(G, A)$). Then P can be extended to a conditional expectation of $C_r^*(G, A, \alpha)$ onto A . In fact, for any state ϕ of A , let $(\pi_\phi, H_\phi, \xi_\phi)$ be the GNS representation of A induced by ϕ and $\eta_\phi = \delta_e \xi_\phi \in L^2(G, H_\phi)$, where $L^2(G, H_\phi)$ is the Hilbert space of all H_ϕ -valued functions η on G such that $\int \|\eta(t)\|^2 dt < \infty$. Denote $\Pi_\phi = \text{Ind } \pi_\phi$ [20, Définition 4.1]. Then $(\Pi_\phi, L^2(G, H_\phi), \eta_\phi)$ is a cyclic $*$ -representation of $L^1(G, A)$, and for each $x \in L^1(G, A)$, $(\Pi_\phi(x)\eta_\phi, \eta_\phi) = \phi(x(e))$ [20, Proposition 4.2(ii)]. It follows that

$$(\Pi_\phi(x^*x)\eta_\phi, \eta_\phi) = \phi((x^*x)(e)) \geq \phi(x(e)^*x(e)) = \phi(P(x)^*P(x)).$$

This implies that $\|P(x)\| \leq \|x\|_r$ (the norm of x in $C_r^*(G, A, \alpha)$) and P extends to a conditional expectation of $C_r^*(G, A, \alpha)$ onto A .

(ii) By the correspondence $a \rightarrow \delta_e a$ ($a \in A$), A is also a C^* -subalgebra of $C^*(G, A, \alpha)$ [20, p. 146]. Since $\|x\|_r \leq \|x\|$ for any $x \in L^1(G, A)$, P in the proof of (i) can be extended to a conditional expectation of $C^*(G, A, \alpha)$ onto A .

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